

The Künneth formula for the twisted de Rham and Higgs cohomologies*

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Abstract

We prove the Künneth formula for the irregular Hodge filtrations on the exponentially twisted de Rham and the Higgs cohomologies of smooth quasi-projective complex varieties. The method involves a careful study of the underlying chain complexes under a certain elimination of indeterminacy.

1 The main statement

Let U be a smooth quasi-projective variety over the field \mathbb{C} of complex numbers, and $f \in \Gamma(U, \mathcal{O}_U)$ a regular function. Attached to such a pair (U, f) and a non-negative integer k , one has the k -th *de Rham cohomology* $H_{\text{dR}}^k(U, f)$ and *Higgs cohomology* $H_{\text{Hig}}^k(U, f)$. The two spaces $H_{\diamond}^k(U, f)$, $\diamond \in \{\text{dR}, \text{Hig}\}$, are of the same finite dimension over \mathbb{C} (see [ESY, Remark 1.3.3]), and are equipped with the decreasing *irregular Hodge filtrations*

$$F^{\lambda} H_{\diamond}^k(U, f) \quad (\lambda \in \mathbb{Q})$$

indexed by the ordered set \mathbb{Q} of rational numbers with finitely many jumps. In the following, we omit the adjective and just call them the *Hodge filtrations*. For the motivations and the basic properties of the Hodge filtration, including in particular the degeneration of the Hodge to de Rham spectral sequence, see [ESY, KKP] and the references therein. We recall the construction in §2.

Now consider two such pairs (U_i, f_i) , $i = 1, 2$. On the product $U := U_1 \times U_2$, consider the regular function f defined by $f(x_1, x_2) := f_1(x_1) + f_2(x_2)$ where $x_i \in U_i$. We call the pair (U, f) the *product* of the two (U_i, f_i) . For $\diamond \in \{\text{dR}, \text{Hig}\}$, there is the canonical map

$$\bigoplus_{i+j=k} H_{\diamond}^i(U_1, f_1) \otimes H_{\diamond}^j(U_2, f_2) \rightarrow H_{\diamond}^k(U, f) \quad (1)$$

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induced by cup product. We equip the space on the left hand side with the product filtration, i.e. the λ -th filtration for $\lambda \in \mathbb{Q}$ is given by the subspace

$$\bigoplus_{i+j=k} \left(\sum_{a+b=\lambda} F^a H_{\diamond}^i(U_1, f_1) \otimes F^b H_{\diamond}^j(U_2, f_2) \right)$$

where the inner sum is taken inside $H_{\diamond}^i(U_1, f_1) \otimes H_{\diamond}^j(U_2, f_2)$. In this article, we prove the following Künneth formula.

Theorem 1.1 *With notations as above, the map (1) is an isomorphism of filtered spaces.*

In particular, denoting $\mathrm{Gr}_F^{\lambda} V$ the λ -th graded piece of a filtered space (V, F) , one has the identification

$$\mathrm{Gr}_F^{\lambda} H_{\diamond}^k(U, f) = \bigoplus_{r,s} \mathrm{Gr}_F^r H_{\diamond}^s(U_1, f_1) \otimes \mathrm{Gr}_F^{\lambda-r} H_{\diamond}^{k-s}(U_2, f_2).$$

The rest of the article begins in §2 with a brief of the construction of the Hodge filtration. We follow the approach of [Y] by putting a filtration on the de Rham complex or the Higgs complex via a certain compactification of the pair (U, f) . Here we introduce the notion of a non-degenerate compactification, which is weaker than that of a good compactification used in [Y], but appears naturally in the later section (see also [S, §4], [SY, §7.3], [M2]). In order to compare the cohomologies with the filtrations of the summands (U_i, f_i) and of their product (U, f) , we construct a particular compactification of (U, f) from the fixed ones of (U_i, f_i) in §3. The proof of the Künneth formula is obtained by a careful investigation of the relations between the involved complexes on the compactifications. In the last §4, we remark that one can interpolate the two spaces $H_{\mathrm{dR}}^k(U, f)$ and $H_{\mathrm{Hig}}^k(U, f)$ as the fibers of the *Kontsevich bundle* on the projective line \mathbb{P}^1 over 1 and 0, respectively. In fact, the fiber over $c \neq 0, \infty$ is equal to $H_{\mathrm{dR}}^k(U, f/c)$ and hence the Künneth formula also holds true. However at ∞ the situation is more complicated in this regard and the direct analogue of the Künneth formula does not hold in general.

2 The filtrations

In this section, we fix a pair (U, f) consisting of a smooth quasi-projective variety U over \mathbb{C} and a regular function $f \in \Gamma(U, \mathcal{O}_U)$.

2.1 The compactification

Let X be a projective variety over \mathbb{C} containing U such that the reduced subvariety $S := X \setminus U$ is a normal crossing divisor. Regard $f \in \Gamma(X, \mathcal{O}_X(*S))$ as a rational function on X . Let P and Z be the pole divisor and the zero divisor of f on X , respectively, and let P_{red} be the support of P .

Definition. (i) We call X a *non-degenerate compactification* of (U, f) if there exists a neighborhood $V \subset X$ of P such that $Z \cap V$ is smooth and $Z + P_{\mathrm{red}}$ forms a reduced normal crossing divisor on V .

(ii) We call X a *good compactification* of (U, f) if f indeed defines a morphism $f : X \rightarrow \mathbb{P}^1$.

The non-degenerate compactification appears in the considerations of rescaling from a good compactification (see [SY, §7.3]), and of Fourier transform (see [S, §4]). It is discussed in [M2] where in this situation, the author calls f *non-degenerate along S* ([M2, Def.2.6]). See also [Y, Prop.4.3] in the case where f is a Laurent polynomial.

If X is non-degenerate, analytically locally at a point of P , there exists a coordinate system

$$\{x_1, \dots, x_\ell, y_1, \dots, y_m, z_1, \dots, z_r\}$$

such that

$$S = (xy) \quad \text{and} \quad f = \frac{1}{x^e} \quad \text{or} \quad f = \frac{z_1}{x^e} \quad (e \in \mathbb{Z}_{>0}^\ell). \quad (2)$$

(Here and after, we use the standard multi-index convention.) If X is good, the second case does not occur.

2.2 The filtered complexes

Fix a non-degenerate compactification X of (U, f) with boundary S . Regard f as a rational function on X and let $P = f^*(\infty)$ be the pole divisor with multiplicities. We have $df \in \Gamma(X, \Omega_X^1(\log S)(P))$.

Definition. (i) The *twisted de Rham complex* and the *Higgs complex* are the complexes

$$(\Omega_X^\bullet(\log S)(*P), \Theta) = \left[\mathcal{O}_X(*P) \xrightarrow{\Theta} \dots \rightarrow \Omega_X^i(\log S)(*P) \xrightarrow{\Theta} \Omega_X^{i+1}(\log S)(*P) \rightarrow \dots \right]$$

where $\Theta = d + df$ and df , respectively. Here df sends a local section ω to $df \wedge \omega$.

(ii) We call the associated hypercohomology groups $\mathbb{H}^k(X, (\Omega_X^\bullet(\log S)(*P), \Theta))$ the *de Rham cohomology* and the *Higgs cohomology of (U, f)* , and denote respectively by

$$H_{\text{dR}}^k(U, f) = H^k(U, d + df) \quad \text{and} \quad H_{\text{Hig}}^k(U, f) = H^k(U, df).$$

(iii) For an effective divisor D on X and $\mu \in \mathbb{Q}$, let

$$\Omega_X^i(\log S)(\lfloor \mu D \rfloor)_+ = \begin{cases} \Omega_X^i(\log S)(\lfloor \mu D \rfloor) & \text{if } \mu \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For $\Theta \in \{d + df, df\}$, $\lambda \in \mathbb{Q}$, let

$$F_X^\lambda(\Theta) = F^\lambda(\Theta) = \left[\mathcal{O}_X(\lfloor -\lambda P \rfloor)_+ \xrightarrow{\Theta} \dots \rightarrow \Omega_X^i(\log S)(\lfloor (i - \lambda)P \rfloor)_+ \xrightarrow{\Theta} \dots \right].$$

The subindex X in $F_X^\lambda(\Theta)$ will be omitted if the base variety is clear. The *Hodge filtrations* of the de Rham and the Higgs cohomologies are

$$F^\lambda H^k(U, \Theta) = \text{Image} \left\{ \mathbb{H}^k(X, F^\lambda(\Theta)) \rightarrow H^k(U, \Theta) \right\} \quad (3)$$

induced by the inclusions of complexes.

(iv) For $\alpha \in \mathbb{Q}$, the *Kontsevich sheaf of differential p -forms* is the coherent subsheaf of $\Omega_X^p(\log S)(*P)$

$$\Omega_f^p(\alpha) = \ker \left\{ \Theta : \Omega_X^p(\log S)(\lfloor \alpha P \rfloor) \rightarrow \Omega_X^{p+1}(*S)/\Omega_X^{p+1}(\log S)(\lfloor \alpha P \rfloor) \right\},$$

and it forms the *Kontsevich complex* $(\Omega_f^\bullet(\alpha), \Theta)$ equipped with the filtration $(\Omega_f^\bullet(\alpha), \Theta)_{\bullet \geq p}$ by direct truncation. We simply write Ω_f^p for $\Omega_f^p(0)$.

Proposition 2.1 (i) For $\alpha \in \mathbb{Q}, p \in \mathbb{Z}$ the \mathcal{O}_X -module $\Omega_f^p(\alpha)$ is locally free of rank $\binom{\dim X}{p}$ with $\Omega_f^p(\alpha) = \Omega_f^p \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lfloor \alpha P \rfloor)$.

(ii) For $\Theta \in \{d + df, df\}$ the three inclusions

$$\begin{aligned} F^0(\Theta) &\rightarrow F^{-\alpha}(\Theta) & 0 \leq \alpha \\ (\Omega_f^\bullet(\alpha), \Theta) &\rightarrow F^{-\alpha}(\Theta) & 0 \leq \alpha \\ (\Omega_f^\bullet(\alpha), \Theta)_{\bullet \geq p} &\rightarrow F^{-\alpha+p}(\Theta) & 0 \leq \alpha < 1 \end{aligned}$$

are quasi-isomorphisms.

Proof. Both statements are local properties for coherent sheaves on X and we can restrict to the coordinates such that (2) holds.

In case $f = \frac{1}{x^e}$, the local freeness and the quasi-isomorphisms are obtained in [ESY, (1.3.1)], [KKP, Lemma 2.12(a)] and [Y, Prop.1.3.], [ESY, Prop.1.4.2], respectively. In fact, the methods are similar to the arguments below.

Consider the second case $f = \frac{z_1}{x^e}$ so that $z_1 \frac{df}{f} = x^e df = dz_1 - z_1 \sum e_i \frac{dx_i}{x_i}$. In this chart, the \mathcal{O}_X -module $\Omega_X^p(\log S)$ is freely generated by

$$z_1 \frac{df}{f} \wedge \bigwedge_{i=1}^{p-1} \xi_i, \quad \{\xi_i\}_{i=1}^{p-1} \subset \left\{ dz_2, \dots, dz_r, \frac{dx_1}{x_1}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m} \right\} \quad (4)$$

and

$$\bigwedge_{i=1}^p \eta_i, \quad \{\eta_i\}_{i=1}^p \subset \left\{ dz_2, \dots, dz_r, \frac{dx_1}{x_1}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m} \right\}. \quad (5)$$

The \mathcal{O}_X -module $\Omega_f^p(\alpha)$ is indeed freely generated by

$$x^{-\lfloor \alpha e \rfloor} \omega_1 \quad \text{and} \quad x^{e - \lfloor \alpha e \rfloor} \omega_2 \quad (6)$$

where ω_1 and ω_2 run through elements in (4) and (5), respectively.

Let E be an irreducible component of P_{red} and

$$A_k^p(\alpha) = \Omega^p(\log S)(\lfloor (\alpha + p)P \rfloor + kE) \quad (p, k \in \mathbb{Z}, \alpha \in \mathbb{Q}).$$

To obtain the quasi-isomorphisms, it suffices to show that the inclusion of complexes

$$(A_{k-1}^\bullet(\alpha), \Theta) \rightarrow (A_k^\bullet(\alpha), \Theta) \quad (7)$$

is a quasi-isomorphism for any k, α . For this, we may assume that $E = (x_1)$ in the local model. Consider local sections $h \in \mathcal{O}_X$, and ω_1 and ω_2 in (4) and (5), respectively. Then $\{h \cdot x^{-[(\alpha+p)e]} x_1^{-k} \omega_i\}$ generates $A_k^p(\alpha)$ and

$$\begin{aligned} \Theta : A_k^p(\alpha)/A_{k-1}^p(\alpha) &\rightarrow A_k^{p+1}(\alpha)/A_{k-1}^{p+1}(\alpha) \\ \frac{h}{x^{[(\alpha+p)e]} x_1^k} \cdot \begin{cases} \omega_1 \\ \omega_2 \end{cases} &\mapsto \begin{cases} 0 \\ \frac{h}{x^{[(\alpha+p+1)e]} x_1^k} (x^e df) \wedge \omega_2. \end{cases} \end{aligned}$$

Therefore the quotient $(A_k^p(\alpha)/A_{k-1}^p(\alpha), \Theta)_{p \in \mathbb{Z}}$ of (7) is exact. \square

Remark. By [SY, Lemma 9.17], the filtered complex $F^\lambda(d + df)$ is quasi-isomorphic to the filtered de Rham complex of the \mathcal{D} -module attached to $(\mathcal{O}_X(*S), d + df)$ with an *irregular Hodge filtration* given in [SY, Def.5.1]. When X is a good compactification, this is discussed in [ESY, Prop.1.7.4].

2.3 The independence

Proposition 2.2 *For $\diamond \in \{\text{dR}, \text{Hig}\}$, the space $H_\diamond^k(U, f)$ with the Hodge filtration is independent of the choice of the non-degenerate compactification of (U, f) . More precisely, if $\pi : X' \rightarrow X$ is a morphism between non-degenerate compactifications extending the identity on U , then there is a natural quasi-isomorphism*

$$R\pi_* F_{X'}^\lambda(\Theta) \simeq F_X^\lambda(\Theta) \quad (\lambda \in \mathbb{Q}). \quad (8)$$

Proof. The assertion for good compactifications is proved in [Y, Thm.1.7], by comparing the degree p components $F_X^\lambda(\Theta)^p$ of $F_X^\lambda(\Theta)$ on various X for a fixed p (and hence the proof works for both $\Theta = d + df$ and df).

In the following we show that by performing successively certain blowups, one can replace a non-degenerate compactification X (thus in the second case of (2)) by a good one and compare the involved chain complexes (c.f. [Y, §4(b)]). Let $\varpi : \tilde{X} \rightarrow X$ be the blowup along the intersection Ξ of $Z = (z_1)$ and the irreducible component (x_1) of P with multiplicity e_1 . Let E be the exceptional divisor, $\tilde{S} = \tilde{X} \setminus U$ and \tilde{P} the pole divisor of $\tilde{f} := \varpi^* f$. We want to establish that $R\varpi_* F_{\tilde{X}}^\lambda(\Theta)$ and $F_X^\lambda(\Theta)$ are canonically quasi-isomorphic.

In case $e_1 > 1$, it can be proved as [Y, Prop.4.4]. In more details, write $\lambda = -\alpha + p$ where $0 \leq \alpha < 1$ and $p \in \mathbb{Z}$. On \tilde{X} define the complex

$$R^\lambda(\Theta) = \left(\Omega_{\tilde{X}}^\bullet(\log \tilde{S})(\lfloor (\alpha - p + \bullet)(\tilde{P} + E) \rfloor)_+, \Theta \right).$$

We have

$$\varpi^* F_X^\lambda(\Theta), F_{\tilde{X}}^\lambda(\Theta) \subset R^\lambda(\Theta).$$

By [Y, Prop.4.4(i)], each component of the complex $R^\lambda(\Theta)/\varpi^* F_X^\lambda(\Theta)$ is a direct sum of copies of $\mathcal{O}_{E/\Xi}(-1)$. Hence the adjunction $F_X^\lambda(\Theta) \rightarrow R\varpi_* R^\lambda(\Theta)$ is a quasi-isomorphism. On the other hand, there are increasing complexes $R_q^\lambda(\Theta)$ on \tilde{X} (those denoted by $R_\lambda(q)$ in [Y, (27)]), which is a complex under either $d + df$ or df with $R_{-1}^\lambda(\Theta) = F_{\tilde{X}}^\lambda(\Theta)$

and $R_{\dim \tilde{X}-p}^\lambda(\Theta) = R^\lambda(\Theta)$ such that $R_q^\lambda(\Theta)/R_{q-1}^\lambda(\Theta)$ is quasi-isomorphic to the complex consisting of a direct sum of copies of $\mathcal{O}_{E/\Xi}(-1)$ concentrated at degree $p+q$. In fact, in [Y, Lemma 4.5], we further introduce the complexes $(K_{\rho,\eta,\xi}^\bullet, d+df)$ and prove the inclusion $(K_{\rho,\eta,\xi}^\bullet, d+df) \subset (K_{\rho,\eta+1,\xi}^\bullet, d+df)$ is a quasi-isomorphism by showing the quotient complex is exact. However on the quotient, one has the equality $d+df = df$ of the differential maps under the condition $e_1 > 1$ and $K_{\rho,\eta,\xi}^\bullet$ is indeed also stable under df . Therefore the proof of [Y, Prop.4.4(ii)] describing $R_q^\lambda(\Theta)/R_{q-1}^\lambda(\Theta)$ goes through in both cases $\Theta = d+df$ and $\Theta = df$ and one obtains that the inclusion $F_{\tilde{X}}^\lambda(\Theta) \subset R^\lambda(\Theta)$ induces a quasi-isomorphism $R\varpi_* F_{\tilde{X}}^\lambda(\Theta) \rightarrow R\varpi_* R^\lambda(\Theta)$.

The following arguments work for any $e_1 \geq 1$ and simplify those in [Y, §4(b)]. (Cf., [KKP, Lemma 2.12(b)].) By Prop.2.1, it suffices to show that for any $0 \leq \alpha < 1, p \in \mathbb{Z}$, the inclusion $\varpi^* \Omega_f^p(\alpha) \subset \Omega_{\tilde{f}}^p(\alpha)$ on \tilde{X} induces an isomorphism $\Omega_f^p(\alpha) \rightarrow R\varpi_* \Omega_{\tilde{f}}^p(\alpha)$ on X . Explicitly the blowup \tilde{X} is defined by the equation $\det \begin{pmatrix} x_1 & z_1 \\ u & v \end{pmatrix} = 0$ in $X \times \mathbb{P}^1$ where $[u : v]$ is the homogeneous coordinate on \mathbb{P}^1 . On the chart $v \neq 0$ with local coordinates

$$\{\bar{u} = u/v, x_2, \dots, x_\ell, y_1, \dots, y_m, z_1, \dots, z_r\},$$

the $\mathcal{O}_{\tilde{X}}$ -module $\Omega_{\tilde{f}}^p(\alpha)$ is generated by the basis

$$\delta \frac{d\tilde{f}}{\tilde{f}} \wedge \bigwedge_{i=1}^{p-1} \xi_i, \quad \{\xi_i\}_{i=1}^{p-1} \subset \left\{ z_1^{-[\alpha(e_1-1)]} \bar{u}^{-[\alpha e_1]} x_2^{-[\alpha e_2]} \dots x_\ell^{-[\alpha e_\ell]} \frac{dz_1}{z_1} + \frac{d\bar{u}}{\bar{u}}, \frac{dx_2}{x_2}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m}, dz_2, \dots, dz_r \right\} \quad (9)$$

and

$$\delta \frac{1}{\tilde{f}} \bigwedge_{i=1}^p \eta_i, \quad \{\eta_i\}_{i=1}^p \subset \left\{ \frac{dz_1}{z_1} + \frac{d\bar{u}}{\bar{u}}, \frac{dx_2}{x_2}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m}, dz_2, \dots, dz_r \right\}. \quad (10)$$

On the chart $v \neq 0$ with local coordinates

$$\{\bar{v} = v/u, x_1, \dots, x_\ell, y_1, \dots, y_m, z_2, \dots, z_r\},$$

the sheaf $\Omega_{\tilde{f}}^p(\alpha)$ is generated by the basis

$$\varepsilon \bar{v} \frac{d\tilde{f}}{\tilde{f}} \wedge \bigwedge_{i=1}^{p-1} \xi_i, \quad \{\xi_i\}_{i=1}^{p-1} \subset \left\{ x_1^{-[\alpha(e_1-1)]} x_2^{-[\alpha e_2]} \dots x_\ell^{-[\alpha e_\ell]} \frac{dx_1}{x_1}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m}, dz_2, \dots, dz_r \right\} \quad (11)$$

and

$$\varepsilon \frac{\bar{v}}{\tilde{f}} \bigwedge_{i=1}^p \eta_i, \quad \{\eta_i\}_{i=1}^p \subset \left\{ \frac{dx_1}{x_1}, \dots, \frac{dx_\ell}{x_\ell}, \frac{dy_1}{y_1}, \dots, \frac{dy_m}{y_m}, dz_2, \dots, dz_r \right\}. \quad (12)$$

On the intersection $u, v \neq 0$, one has $\frac{dx_1}{x_1} = \frac{dz_1}{z_1} + \frac{d\bar{u}}{\bar{u}}$. Using the basis (6) of $\Omega_f^p(\alpha)$, the $\mathcal{O}_{E/\Xi}$ -module $\Omega_{\tilde{f}}^p(\alpha)/\varpi^* \Omega_f^p(\alpha)$ equals either zero if $[\alpha(e_1-1)] \neq [\alpha e_1]$ or otherwise $\binom{\ell+m+r}{p}$ copies of $\mathcal{O}_{E/\Xi}(-1)$ generated by (9, 10) and (11, 12) on the two charts, respectively. Therefore we obtain that $F_{\tilde{X}}^\lambda(\Theta)$ and $R\varpi_* F_{\tilde{X}}^\lambda(\Theta)$ are naturally quasi-isomorphic.

One then iteratively takes the blowups along the intersections of irreducible components of the zero and the pole divisors as in [Y, §4(b)] (the diagram (26) therein) to obtain a good compactification X' of (U, f) from the non-degenerate X with the canonical quasi-isomorphism (8). \square

Remark. (i) By the E_1 -degeneration [ESY, Thm.1.2.2], [M1, Thm.1.1] on a good compactification (see also [KKP, Theorem 2.18]), we conclude that the arrow in (3) is injective for any non-degenerate X and indices k, λ .

(ii) We have $H_{\text{dR}}^k(U, f) = \mathbb{H}^k(X, (\Omega_X^\bullet(*S), d + df))$ by the arguments of [Y, Cor.1.4]. On the other hand, $H_{\text{Hig}}^k(U, f) \neq \mathbb{H}^k(X, (\Omega_X^\bullet(*S), df))$ in general which can be seen by considering the case U affine and $f = 0$.

3 The proof of the main result

We begin with two pairs $(U_i, f_i), i = 1, 2$, and their product (U, f) . Fix good compactifications X_i of (U_i, f_i) such that $S_i := X_i \setminus U_i$ are strict normal crossing divisors. The proof of Thm.1.1 consists of two steps. In the first step §3.1, we construct explicitly a non-degenerate compactification X of (U, f) from $X_1 \times X_2$ by successive blowups. In step two §3.2, we compare the filtered de Rham or the Higgs complex on X , which gives the Hodge filtration on $H_\diamond^k(U, f)$, with a certain filtered complex on $X_1 \times X_2$ that gives the product filtration using the explicit construction of X .

3.1 An explicit elimination

For each $i = 1, 2$, take an open covering of X_i with a system of local coordinates

$$\{x_{i,1}, \dots, x_{i,\ell_i}, x_{i,\ell_i+1}, \dots, x_{i,\ell_i+m_i}, x_{i,\ell_i+m_i+1}, \dots, x_{i,\ell_i+m_i+r_i}\}$$

such that

$$f_i = \frac{1}{x_{i,1}^{e_{i,1}} \cdots x_{i,\ell_i}^{e_{i,\ell_i}}} \quad (e_{i,j} > 0), \quad S_i = (x_{i,1} \cdots x_{i,\ell_i+m_i}).$$

As for the initial data in the inductive construction, we consider the compactification $X_1 \times X_2$ of U with the systems of local coordinates $\{x_{i,j}\}$.

Suppose we have constructed a compactification Y of U and its systems of local coordinates

$$\{y_1, \dots, y_\ell, y_{\ell+1}, \dots, y_{\ell+m}, y_{\ell+m+1}, \dots, y_{\ell+m+r}\}, \quad (13)$$

together with a birational map $\pi : Y \rightarrow X_1 \times X_2$ such that

$$\pi^* f_1 = \frac{1}{y_1^{a_1} \cdots y_\ell^{a_\ell}}, \quad \pi^* f_2 = \frac{1}{y_1^{b_1} \cdots y_\ell^{b_\ell}}, \quad Y \setminus U = (y_1 \cdots y_{\ell+m}) \quad (14)$$

for some $a, b \geq 0$ with $b_i > 0$ if $a_i = 0$. Let T be the boundary divisor $Y \setminus U$. For a pair of irreducible components D_1, D_2 of T , set

$$\Delta_Y(D_1, D_2) = \begin{cases} \prod_{i=1}^2 (\text{ord}_{D_i}(\pi^* f_1) - \text{ord}_{D_i}(\pi^* f_2)) & \text{if } D_1 \neq D_2, D_1 \cap D_2 \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $\Delta_Y(D_1, D_2) \geq 0$ for any pair (D_1, D_2) , i.e., $a \geq b$ or $b \geq a$ in terms of the systems of local coordinates as above, then the zero divisor of π^*f is smooth in a neighborhood of T and intersects T transversally. Thus Y is a non-degenerate compactification of (U, f) .

Otherwise, pick a pair (D_1, D_2) such that $\Delta_Y(D_1, D_2) < 0$ and is the smallest among all possible values of Δ_Y . Let \tilde{Y} be the blowup of Y along $D_1 \cap D_2$ and $\tilde{\pi} : \tilde{Y} \rightarrow X_1 \times X_2$ the induced map. Then $\tilde{T} := \tilde{Y} \setminus U$ consists of the exceptional divisor E and $\{\tilde{D}\}$ where \tilde{D} denotes the proper transform of an irreducible component D of T . To construct the explicit systems of local coordinates of \tilde{Y} , we may assume that $D_i = (y_i)$ for $i = 1, 2$ with $a_1 > b_1$ and $a_2 < b_2$ after rearrangement. The blowup is defined by the equation $y_1v = y_2u$ where $[u : v]$ is the homogeneous coordinate of \mathbb{P}^1 . In this chart of coordinates of \tilde{Y} , we add two charts to \tilde{Y} (and away from the blowup center, we pass the charts of Y to \tilde{Y}). In the chart $v \neq 0$ of \mathbb{P}^1 , we consider the local coordinates

$$\{\bar{u} := u/v, y_2, y_3, \dots, y_{\ell+m+r}\}. \quad (15)$$

Then

$$\begin{aligned} \tilde{\pi}^*f_1 &= \frac{1}{\bar{u}^{a_1} y_2^{a_1+a_2} y_3^{a_3} \dots y_{\ell}^{a_{\ell}}}, & \tilde{\pi}^*f_2 &= \frac{1}{\bar{u}^{b_1} y_2^{b_1+b_2} y_3^{b_3} \dots y_{\ell}^{b_{\ell}}}, \\ \tilde{T} &= (\bar{u} y_2 \dots y_{\ell+m}), & \tilde{D}_1 &= (\bar{u}), \quad E = (y_2). \end{aligned} \quad (16)$$

In the chart $u \neq 0$ of \mathbb{P}^1 , we consider the local coordinates

$$\{\bar{v} := v/u, y_1, y_3, \dots, y_{\ell+m+r}\}. \quad (17)$$

Then

$$\begin{aligned} \tilde{\pi}^*f_1 &= \frac{1}{\bar{v}^{a_2} y_1^{a_1+a_2} y_3^{a_3} \dots y_{\ell}^{a_{\ell}}}, & \tilde{\pi}^*f_2 &= \frac{1}{\bar{v}^{b_2} y_1^{b_1+b_2} y_3^{b_3} \dots y_{\ell}^{b_{\ell}}}, \\ \tilde{T} &= (\bar{v} y_1 y_3 \dots y_{\ell+m}), & \tilde{D}_2 &= (\bar{v}), \quad E = (y_1). \end{aligned} \quad (18)$$

One has $\tilde{D}_1 \cap \tilde{D}_2 = \emptyset$ and

$$\begin{aligned} \Delta_{\tilde{Y}}(\tilde{D}, \tilde{D}') &= \Delta_Y(D, D') \quad \text{for } (D, D') \neq (D_1, D_2), (D_2, D_1) \\ \Delta_{\tilde{Y}}(\tilde{D}_i, E) &= \Delta_Y(D_1, D_2) + (\text{ord}_{D_i}(\pi^*f_1) - \text{ord}_{D_i}(\pi^*f_2))^2 > \Delta_Y(D_1, D_2) \\ \Delta_{\tilde{Y}}(\tilde{D}, E) &= \Delta_Y(D, D_1) + \Delta_Y(D, D_2). \end{aligned}$$

Notice that

$$\Delta_Y(D, D_i) = (\text{ord}_D(\pi^*f_1) - \text{ord}_D(\pi^*f_2)) \cdot (-a_i + b_i)$$

and consequently

$$\Delta_{\tilde{Y}}(\tilde{D}, E) \begin{cases} > \min\{\Delta_Y(D, D_1), \Delta_Y(D, D_2)\} & \text{if } \text{ord}_D(\pi^*f_1) \neq \text{ord}_D(\pi^*f_2) \\ = 0 & \text{if } \text{ord}_D(\pi^*f_1) = \text{ord}_D(\pi^*f_2). \end{cases}$$

We then replace Y with its systems of local coordinates by \tilde{Y} with the coordinates constructed above.

Observe that after a finite number of blowups in this procedure, the smallest possible value of the function Δ , if it is negative, strictly increases. Hence repeating this construction, it produces a non-degenerate compactification X of (U, f) obtained by a sequence $X \rightarrow \dots \rightarrow X_1 \times X_2$ of explicit blowups.

3.2 Relations between complexes

We shall put a filtered complex on each step of the sequence of blowups constructed in §3.1 and compare them under push-forwards.

Consider a birational map $\pi : Y \rightarrow X_1 \times X_2$ and let P_{Y,f_j} be the pole divisor of the pullback of f_j on Y . For any $\lambda \in \mathbb{Q}$, consider the subsheaf $\mathcal{O}_Y^{(\lambda)}$ of $\mathcal{O}_Y(* (P_{Y,f_1} + P_{Y,f_2}))$ given by

$$\mathcal{O}_Y^{(\lambda)} = \sum_{0 \leq \theta \leq 1} \mathcal{O}_Y(\lfloor \lambda((1-\theta)P_{Y,f_1} + \theta P_{Y,f_2}) \rfloor),$$

which is coherent but not locally free in general. Let $T = Y \setminus \pi^{-1}(U)$ and

$$\Omega_Y^{p,(\lambda)} = \mathcal{O}_Y^{(\lambda)} \otimes_{\mathcal{O}_Y} \Omega_Y^p(\log T).$$

For $\Theta \in \{d + df, df\}$, consider the complex on Y

$$F_Y^{(\lambda)}(\Theta) = \left[\mathcal{O}_Y^{(-\lambda)+} \xrightarrow{\Theta} \Omega_Y^{1,(1-\lambda)+} \xrightarrow{\Theta} \Omega_Y^{2,(2-\lambda)+} \rightarrow \dots \right]$$

where

$$\Omega_Y^{p,(p-\lambda)+} = \begin{cases} \Omega_Y^{p,(p-\lambda)} & \text{if } p - \lambda \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(It is indeed a sub-complex of $(\Omega_Y^\bullet(*T), \Theta)$.)

At the final step $Y = X$ of the blowups, X is a non-degenerate compactification of (U, f) and $F_Y^{(\lambda)}(\Theta)$ is indeed the filtration defining the desired Hodge filtration. The following two lemmas describe the situations in the initial step $Y = X_1 \times X_2$ and in each blowup $\varpi : \tilde{Y} \rightarrow Y$ appeared in the sequence occurred in §3.1, respectively.

Lemma 3.1 *Consider $Y = X_1 \times X_2$ and let $F_{\boxtimes}^\lambda(\Theta)$ be the product filtration of $F_{X_i}^\lambda(\Theta_i)$ whose p -th component is*

$$F_{\boxtimes}^\lambda(\Theta)^p = \sum_{a \in \mathbb{Q}, q \in \mathbb{Z}} F_{X_1}^a(\Theta_1)^q \boxtimes F_{X_2}^{\lambda-a}(\Theta_2)^{p-q}$$

inside $\Omega_Y^p(*T)$. (Here $\Theta_i = d + df_i$ and df_i if $\Theta = d + df$ and df , respectively.)

(i) The filtration $F_Y^{(\lambda)}(\Theta)$ coincides with $F_{\boxtimes}^\lambda(\Theta)$.

(ii) One has

$$\mathbb{H}^k(Y, F_{\boxtimes}^\lambda(\Theta)) = \bigoplus_{i+j=k} \left(\sum_{a+b=\lambda} \mathbb{H}^i(X_1, F_{X_1}^a(\Theta_1)) \otimes \mathbb{H}^j(X_2, F_{X_2}^b(\Theta_2)) \right) \quad (19)$$

where the inner sum is taken inside the vector space $\mathbb{H}^i(U_1, \Theta_1) \otimes \mathbb{H}^j(U_2, \Theta_2)$.

Proof. (i) Indeed the inclusion $F_{\boxtimes}^{\lambda}(\Theta) \subset F_Y^{(\lambda)}(\Theta)$ is clear. Conversely suppose that $\mu := p - \lambda > 0$. The p -th degree component $\Omega_Y^{p,(\mu)}$ of $F_Y^{(\lambda)}(\Theta)$ is generated by products of elements in

$$\Omega_{X_1}^q(\log S_1)(\lfloor (1-\theta)\mu P_{X_1} \rfloor) \quad \text{and} \quad \Omega_{X_2}^{p-q}(\log S_2)(\lfloor \theta\mu P_{X_2} \rfloor),$$

which are the q -th and the $(p-q)$ -th components of $F_{X_1}^{q-(1-\theta)\mu}(\Theta_1)$ and $F_{X_2}^{p-q-\theta\mu}(\Theta_2)$, respectively. The latter two contribute to $F_{\boxtimes}^{\lambda}(\Theta)$.

(ii) We have the natural external products

$$F_{X_1}^a(\Theta_1) \boxtimes F_{X_2}^b(\Theta_2) \rightarrow F_{\boxtimes}^{\lambda}(\Theta)$$

for all $a + b = \lambda$. Using Čech resolution or representatives in smooth forms, one obtains the cup product

$$\mathbb{H}^i(X_1, F_{X_1}^a(\Theta_1)) \otimes \mathbb{H}^j(X_2, F_{X_2}^b(\Theta_2)) \rightarrow \mathbb{H}^{i+j}(Y, F_{X_1}^a(\Theta_1) \boxtimes F_{X_2}^b(\Theta_2)).$$

One the other hand, one has from the definition that

$$\mathrm{Gr}_{F_{\boxtimes}^{\lambda}(\Theta)}^{\lambda} = \bigoplus_{a+b=\lambda} \mathrm{Gr}_{F_{X_1}}^a(\Theta_1) \boxtimes \mathrm{Gr}_{F_{X_2}}^b(\Theta_2).$$

Again there is the cup product

$$\bigoplus_{i+j=k} \mathbb{H}^i(X_1, \mathrm{Gr}_{F_{X_1}}^a(\Theta_1)) \otimes \mathbb{H}^j(X_2, \mathrm{Gr}_{F_{X_2}}^b(\Theta_2)) \rightarrow \mathbb{H}^k(Y, \mathrm{Gr}_{F_{X_1}}^a(\Theta_1) \boxtimes \mathrm{Gr}_{F_{X_2}}^b(\Theta_2)).$$

Therefore to complete the assertion, it suffices to show that the arrow above is an isomorphism, which can be obtained by directly truncating the involved complexes and inductively using the Künneth formula for coherent sheaves [SW, Thm.1]. \square

Lemma 3.2 *Let $\varpi : \tilde{Y} \rightarrow Y$ be the blowup constructed in §3.1. Then the pullback $\varpi^* F_Y^{(\lambda)}(\Theta)$ is a sub-complex of $F_{\tilde{Y}}^{(\lambda)}(\Theta)$ and the following hold.*

(i) *Each component $F_{\tilde{Y}}^{(\lambda)}(\Theta)^p / \varpi^* F_Y^{(\lambda)}(\Theta)^p$ of the quotient is supported on the exceptional divisor E and is a direct sum of copies of relative $\mathcal{O}(-1)$ of the \mathbb{P}^1 -bundle E over the center of the blowup ϖ .*

(ii) *We have the canonical quasi-isomorphism $F_Y^{(\lambda)}(\Theta) \rightarrow R\varpi_* F_{\tilde{Y}}^{(\lambda)}(\Theta)$.*

Proof. (i) Let $\tilde{T} = \tilde{Y} \setminus U$. Again assume that $\mu := p - \lambda \geq 0$. Since $\varpi^* \Omega_Y^p(\log T) = \Omega_{\tilde{Y}}^p(\log \tilde{T})$, we only need to consider the difference between $\mathcal{O}_{\tilde{Y}}^{(\mu)}$ and $\varpi^* \mathcal{O}_Y^{(\mu)}$. We use the local coordinates (13) with the properties (14). Then $\mathcal{O}_Y^{(\mu)}$ is generated by various

$$g := \frac{1}{y^{\lfloor c \rfloor}} \tag{20}$$

where $c = (1 - \theta)\mu a + \theta\mu b$ for $0 \leq \theta \leq 1$. On the other hand, $\mathcal{O}_{\tilde{Y}}^{(\mu)}$ is generated by various

$$h_{\bar{u}} := \frac{1}{\bar{u}^{[c_1]} y_2^{[c_1+c_2]} y_3^{[c_3]} \dots y_\ell^{[c_\ell]}} \quad \text{and} \quad h_{\bar{v}} := \frac{1}{\bar{v}^{[c_2]} y_1^{[c_1+c_2]} y_3^{[c_3]} \dots y_\ell^{[c_\ell]}}$$

on the two charts (15) and (17) satisfying (16) and (18), respectively. We have

$$\langle h_{\bar{u}} \rangle_{\mathcal{O}_{\tilde{Y}}} / \varpi^* \langle g \rangle_{\mathcal{O}_Y} = \begin{cases} \langle h_{\bar{u}} \rangle_{\mathcal{O}_E} & \text{if } [c_1 + c_2] = [c_1] + [c_2] + 1 \\ 0 & \text{if } [c_1 + c_2] = [c_1] + [c_2] \end{cases}$$

and similarly on the chart $u \neq 0$. Observe on the intersection that $h_{\bar{u}} = \bar{v}^{-1} h_{\bar{v}}$ if $[c_1 + c_2] \neq [c_1] + [c_2]$. Thus $\mathcal{O}_{\tilde{Y}}^{(\mu)} / \varpi^* \mathcal{O}_Y^{(\mu)}$ is a direct sum of copies of the relative $\mathcal{O}(-1)$.

(ii) There are only finitely many $y^{-[c]}$ occurred in the generators in (20); call the appeared (distinct) monomials ξ_1, \dots, ξ_k ordered by the increment of the corresponding parameter θ . Consider locally the filtration $M_i = \langle \xi_1, \dots, \xi_i \rangle$ of \mathcal{O}_Y -submodules of $\mathcal{O}_Y^{(\mu)}$. On this chart one has the short exact sequence

$$\begin{aligned} 0 \rightarrow N_{i+1} \rightarrow M_i \oplus \langle \xi_{i+1} \rangle_{\mathcal{O}_Y} &\rightarrow M_{i+1} \rightarrow 0 \\ (\omega, \eta) &\mapsto \omega - \eta. \end{aligned}$$

Here if $\xi_i = \prod y_j^{-c_{i,j}}$, then $c_{i,j}$ is monotone as a function of i and $N_{i+1} = M_i \cap \langle \xi_{i+1} \rangle$ is the invertible sheaf generated by

$$\prod y_j^{-\min\{c_{i,j}, c_{i+1,j}\}}$$

(with diagonal embedding to $M_i \oplus \langle \xi_{i+1} \rangle_{\mathcal{O}_Y}$). Applying inductively the projection formula to the locally free $\langle \xi_i \rangle_{\mathcal{O}_Y}$ and N_i , one obtains that

$$R\varpi_* \varpi^* M_i = R\varpi_*(\mathcal{O}_{\tilde{Y}} \otimes \varpi^* M_i) = R\varpi_* \mathcal{O}_{\tilde{Y}} \otimes M_i = M_i$$

since $R\varpi_* \mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y$ for the birational morphism ϖ . Together with the computation in (i), the assertion follows. \square

The proof of the main theorem 1.1 is now completed by the above two lemmas.

4 The Brieskorn lattice

We indicate that the Künneth formula for $H_{\text{dR}}^k(U, f)$ and $H_{\text{Hig}}^k(U, f)$ can be put together into a family version (cf., [SY, §1.3, §6.2], [KKP, §3.2] and [M1]).

Consider the affine line \mathbb{A}_u^1 with a fixed coordinate u .

Fix a pair (U, f) and a non-degenerate compactification X . Let $\pi : X \times \mathbb{A}_u^1 \rightarrow \mathbb{A}_u^1$ be the projection. The k -th Brieskorn lattice of (U, f) (cf., [SY, §6.1]) is the coherent sheaf on \mathbb{A}_u^1

$$\mathcal{G}^k(U, f) := R^k \pi_* \left(\Omega_f^q(\alpha) \boxtimes \mathcal{O}_{\mathbb{A}_u^1}, u \cdot d_X + df \right)_{q \geq 0}.$$

Here d_X is the derivative with respect to the component X only.

Let

$$F^{-\alpha+p}\mathcal{G}^k(U, f) := R^k\pi_* \left(\Omega_f^q(\alpha) \boxtimes \mathcal{O}_{\mathbb{A}_u^1}, u \cdot d_X + df \right)_{q \geq p} \quad (0 \leq \alpha < 1).$$

According to the quasi-isomorphisms in Prop.2.1(ii) and the E_1 -degeneration

$$\dim H^k(U, \Theta) = \sum_q \dim H^{k-q} \left(X, \Omega_f^q(\alpha) \right) \quad (\Theta \in \{d + df, df\}),$$

the $\mathcal{O}_{\mathbb{A}_u^1}$ -module $\mathcal{G}^k(U, f)$ is free and independent of the choice of α , and the canonical maps $F^{-\alpha+p}\mathcal{G}^k(U, f) \rightarrow \mathcal{G}^k(U, f)$ indeed define a filtration by free subsheaves of $\mathcal{G}^k(U, f)$ with free quotients whose fibers are

$$F^{-\alpha+p}\mathcal{G}^k(U, f)|_{u=c} = \begin{cases} F^{-\alpha+p}H_{\text{Hig}}^k(U, f) & c = 0 \\ F^{-\alpha+p}H_{\text{dR}}^k(U, f/c) & c \neq 0 \end{cases} \quad (21)$$

under the base-change map (cf. the arguments in the proof of [ESY, Prop.1.5.1]).

Now consider as in §3 two pairs (U_i, f_i) and their product (U, f) . We again have the natural map

$$\bigoplus_{0 \leq i \leq k} \mathcal{G}^i(U_1, f_1) \otimes_{\mathcal{O}_{\mathbb{A}_u^1}} \mathcal{G}^{k-i}(U_2, f_2) \rightarrow \mathcal{G}^k(U, f)$$

obtained by cup product. Similar to the proof of [ESY, Prop.1.5.1], the fiber-wise Künneth formula Thm.1.1 shows that the above map is an isomorphism strictly compatible with the filtrations.

On the other hand, fix $0 \leq \alpha < 1$. One can naturally complete $(\mathcal{G}^k(U, f), F^{-\alpha+p})_{p \in \mathbb{Z}}$ into a filtered bundle on \mathbb{P}^1 by adding the filtered cohomology space

$$F^p H_{d, \alpha}^k(U, f) := \text{Image} \left\{ \mathbb{H}^k \left(X, (\Omega_f^\bullet(\alpha), d)_{\bullet \geq p} \right) \rightarrow \mathbb{H}^k \left(X, (\Omega_f^\bullet(\alpha), d) \right) \right\}$$

as the fiber over $u = \infty$. The resulting sheaf on \mathbb{P}^1 is called the *Kontsevich bundle* and denoted by $\mathcal{K}_\alpha^k(U, f)$. The filtered space $H_{d, \alpha}^k(U, f)$ depends on α but does not depend on the choice of the non-degenerate compactification X since in fact ([SY, Thm.1.11(a), §1.3], [M1, Thm.1.2(i,ii)])

- (i) the bundle $\mathcal{K}_\alpha^k(U, f)$ can be obtained (as a Deligne extension) by using a natural algebraic connection on $\mathcal{G}^k(U, f)$ (see [SY, Lemma 6.2] with the aid of Prop.2.1(ii) in the case of non-degenerate compactification X) with regular singularity at $u = \infty$, and
- (ii) under the base-change, one has

$$\left(H_{d, \alpha}^k(U, f), F^\bullet \right) = \left(\mathcal{K}_\alpha^k(U, f), \text{HN}^\bullet \right)_{|u=\infty}$$

where $\text{HN}^p \mathcal{K}_\alpha^k(U, f)$ is the Harder-Narasimhan filtration on the locally free sheaf $\mathcal{K}_\alpha^k(U, f)$ normalized with Gr_{HN}^p isomorphic to a direct sum of copies of $\mathcal{O}(p)$ on \mathbb{P}^1 .

However, one does not have the direct analogue of the Künneth formula for $H_{d,\alpha}^\bullet$ in general. For example, in the case $(U_i, f_i) = (\mathbb{A}^1, x^2)$ where x is a global coordinate on the affine line \mathbb{A}^1 , one has

$$\dim H_{d,0}^k(U_i, f_i) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the cup product

$$H_{d,0}^1(U_1, f_1) \otimes H_{d,0}^1(U_2, f_2) \rightarrow H_{d,0}^2(U, f)$$

is zero.

References

- [ESY] H. Esnault, C. Sabbah and J.-D. Yu, E_1 -degeneration of the irregular Hodge filtration (with an appendix by Morihiko Saito). To be published in *J. Reine Angew. Math.* DOI: 10.1515/crelle-2014-0118.
- [KKP] L. Katzarkov, M. Kontsevich and T. Pantev, Bogomolov-Tian-Todorov theorems for Landau-Ginzburg models. Preprint arXiv:1409.5996v2.
- [M1] T. Mochizuki, A twistor approach to the Kontsevich complexes. Preprint arXiv:1501.04145v1.
- [M2] T. Mochizuki, Twistor property of GKZ-hypergeometric systems. Preprint arXiv:1501.04146v2.
- [S] C. Sabbah, Monodromy at infinity and Fourier transform. *Publ. Res. Inst. Math. Sci.* 33 (1997), no. 4, 643-685.
- [SY] C. Sabbah and J.-D. Yu, On the irregular Hodge filtration of exponentially twisted mixed Hodge modules. *Forum Math. Sigma* 3 (2015), e9, 71 pp.
- [SW] J.H. Sampson and G. Washnitzer, A Künneth formula for coherent algebraic sheaves. *Illinois J. Math.* 3 (1959), no. 3, 389-402.
- [Y] J.-D. Yu, Irregular Hodge filtration on twisted de Rham cohomology. *Manuscripta Math.* 144 (2014), no. 1, 99-133.